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The instability of inviscid jets and wakes in compressible fluid

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The instability of a two-dimensional jet with respect to three-dimensional disturbances and that of an axially symmetric jet with respect to azimuthally periodic disturbances are studied, for the inviscid flow of a compressible fluid. In both cases the undisturbed velocity is assumed to be uniform in the jet. It is shown analytically that a two-dimensional jet is unstable under small disturbances, either subsonic or supersonic. There is no upper limit in Mach number, as was found for a plane vortex sheet, above which the flow is completely stable. Numerical calculations for the eigenvalues for both the two-dimensional jet and the axially symmetric jet have been made. The results indicate that the increase of Mach number tends to stabilize the flow. For the two-dimensional jet, the larger the angle between the direction of wave propagation and that of the main flow, the more the flow will be destabilized. For the axially symmetric jet, the flow is more unstable under azimuthally periodic disturbances than under rotationally symmetric ones, at small wave number.

1. Introduction

Under certain circumstances, the central region of a high-speed jet may be such that the velocity is nearly constant, changing rapidly to the free stream velocity in a narrow boundary region. In order to increase the understanding of stability theory of high-speed jets of a viscous compressible fluid, it appears useful to investigate the stability of jet flows of an inviscid fluid with the previously mentioned profile. The stability of a tangential discontinuity has been considered by many authors for both incompressible and compressible fluids. For compressible flow, the stability of a plane vortex sheet has been considered by Landau (1944), Hatanaka (1947), Pai (1954) and Miles (1958). Fejer & Miles (1963) infer that a generalization of the two-dimensional disturbance considered by Miles (1958) provides an extension of the stability criterion to three-dimensional disturbances. Pai and Miles also derived eigenvalue equations for the stability of a two-dimensional jet of an inviscid fluid under two-dimensional disturbances in their treatments of a plane vortex sheet. For incompressible flow, the axially symmetric jet of an inviscid fluid has been treated by Batchelor & Gill (1962).

It is our purpose in this paper to carry on the further investigation on the stability of the two-dimensional jet, and to study the stability of the axially symmetric jet of an inviscid compressible fluid.

Until now, no rigorous evidence has been given to support the existence of supersonic disturbances, i.e. disturbances for which the wave speed is supersonic relative to the local flow. Studies of supersonic disturbances have been restricted by the non-diminishing disturbance amplitude at infinity. In most cases it has been assumed that these disturbances are insignificant, because no discrete eigenvalue problem exists for them (pp. 70–71, Lin 1955). It should also be noted that Lin (1953) considered the effect of a limitation to subsonic disturbances as leading immediately to some criterion for stability and further noted that three-dimensional disturbances should be carefully considered. Miles pointed out that supersonic disturbances may cause instability for a plane vortex sheet; an interpretation could be inferred from Sommerfeld's finiteness and radiation conditions. This may indeed explain the significance of supersonic disturbances. Although the radiation condition in a stability investigation has not met with universal acceptance, Miles's remark may explain the possible significance of supersonic disturbances.

The significance of supersonic disturbances in a finite region will be emphasized in the present paper. In a jet flow, we can expect a situation such that disturbances are supersonic relative to the finite flow region of the jet; and, after they go across the discontinuous layer of the main flow, they become subsonic relative to the surrounding free stream. The outer boundary condition is finite in this case. It will be shown later (§ 2.3) that a neutral supersonic disturbance relative to the jet can exist in an inviscid jet.

2. Two-dimensional jet

2.1. *Disturbance equations*

For an inviscid jet, either two-dimensional or axially symmetric, discontinuities in velocity and temperature distributions occur between the jet and the free stream. The thin layer of discontinuity is considered as a vortex sheet with infinite vorticity. Thus, for a two-dimensional jet, we have two parallel plane vortex sheets; for an axially symmetric jet, we have a cylindrical vortex sheet. Here we shall assume that the primary flow is parallel and irrotational, except at the plane or cylindrical vortex sheets; thus, the velocity components can be written as follows:

$$u = U + u', \quad v = v', \quad w = w', \tag{1}$$

where U is the constant main flow velocity, and 'primed' quantities are disturbance velocities.

Three-dimensional disturbances are considered in our investigation of the two-dimensional jet; therefore, all disturbance quantities are functions of x, y, z and t. Under small disturbances the linearized inviscid disturbance equations for a compressible fluid with constant specific heats are given by

$$\text{continuity} \qquad \frac{\partial \rho'}{\partial t} + U \frac{\partial \rho'}{\partial x} = -\bar{\rho}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right), \tag{2}$$

momentum
$$\bar{\rho}\left(\frac{\partial u'}{\partial t}+U\frac{\partial u'}{\partial x}\right) = -\frac{\partial p'}{\partial x}, \tag{3}$$
$$\bar{\rho}\left(\frac{\partial v'}{\partial t}+U\frac{\partial v'}{\partial x}\right) = -\frac{\partial p'}{\partial y}, \tag{4}$$
$$\bar{\rho}\left(\frac{\partial w'}{\partial t}+U\frac{\partial w'}{\partial x}\right) = -\frac{\partial p'}{\partial z}, \tag{5}$$
energy
$$\bar{\rho}c_v\left(\frac{\partial T'}{\partial t}+U\frac{\partial T'}{\partial x}\right) = -\bar{p}\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right), \tag{6}$$
state
$$p'/\bar{p} = \rho'/\bar{\rho}+T'/\bar{T}, \tag{7}$$
where ρ is the density, p the pressure and T the temperature of the flow.

Consider the disturbance to be an oblique plane wave propagating at an angle Θ with respect to the x-direction. The dimensional disturbance quantities can be expressed as

$$\left.\begin{aligned} u',v',w' &= \{f(y),g(y),h(y)\}\exp\left[i(\alpha x+\beta z-\alpha ct)\right],\\ p',T',\rho' &= \{\pi(y),\theta(y),m(y)\}\exp\{i(\alpha x+\beta z-\alpha ct)],\end{aligned}\right\} \tag{8}$$

where
$$c = c_r+ic_i,$$

α is the wave-number in x-direction, and β is the wave-number in z-direction.

The angle Θ is obtained from the relation
$$\Theta = \arccos\left[\alpha(\alpha^2+\beta^2)^{-\frac{1}{2}}\right]. \tag{9}$$

The real part of c in equation (8) is the wave propagation velocity in the x-direction; the imaginary part of c indicates whether the disturbance is amplified, neutral or damped, according to whether c_i is positive, zero or negative.

By introducing the above disturbances, the disturbance equations become:†

continuity
$$i\alpha(U-c)\,m = -\bar{\rho}(g'+i\alpha f+i\beta h), \tag{10}$$
momentum
$$\bar{\rho}(U-c)f = -\pi, \tag{11}$$
$$i\alpha\bar{\rho}(U-c)\,g = -\pi', \tag{12}$$
$$\alpha\bar{\rho}(U-c)\,h = -\beta\pi, \tag{13}$$
energy
$$i\alpha(U-c)\,\theta = (\gamma-1)\,\bar{T}(g'+i\alpha f+i\beta h), \tag{14}$$
state
$$\pi/\bar{p} = m/\bar{\rho}+\theta/\bar{T}. \tag{15}$$

2.2. *Eigenvalue equation*

Elimination of m and θ in (10), (14) and (15) gives
$$\pi = \gamma\bar{p}(ig'/\alpha-f-\beta h/\alpha)/(U-c). \tag{16}$$

Substitution of f g and h obtained from (11), (12) and (13) yields the differential equation for the pressure disturbance:
$$\pi''-\{1-[(U-c)/a]^2\cos^2\Theta\}(\alpha^2+\beta^2)\,\pi = 0, \tag{17}$$

† 'Prime' is used to denote differentiation with respect to y in the rest of this section, unless otherwise specified.

where a is the local velocity of sound. Therefore, the exact solution for π is

$$\pi = K_1 \exp(\tilde{\alpha}\xi y) + K_2 \exp(-\tilde{\alpha}\xi y), \tag{18}$$

where K_1 and K_2 are constants to be determined, and $\tilde{\alpha}$ and ξ are given as follows:

$$\tilde{\alpha} = \sqrt{(\alpha^2+\beta^2)}, \tag{19}$$

$$\xi = \sqrt{\left[1-\left(\frac{U-c}{a}\right)^2 \cos^2\Theta\right]}. \tag{20}$$

The boundary conditions for the symmetrical pressure disturbance are

$$\pi = 0 \quad \text{at} \quad y = 0, \quad \pi' = 0 \quad \text{at} \quad y = \infty, \tag{21}$$

and those for the antisymmetrical pressure disturbance are

$$\pi' = 0 \quad \text{at} \quad y = 0, \infty. \tag{22}$$

Consider the upper half of the jet. Let the half width of the jet be l. Because π is bounded at infinity, K_1 in equation (18) must be zero for $y > l$. Assume that the flow is isoenergetic; therefore, the above treatments hold for both inside and outside of the jet. By introducing the condition that pressures on both sides of the vortex sheet must be equal, it follows that

$$\left.\begin{aligned} \pi_- &= K_1[\exp(\tilde{\alpha}\xi_- y) - \exp(-\tilde{\alpha}\xi_- y)] \quad \text{for} \quad 0 < y < l, \\ \pi_+ &= K_1[\exp(\tilde{\alpha}\xi_- l) - \exp(-\tilde{\alpha}\xi_- l)]\exp[-\tilde{\alpha}\xi_+(y-l)] \quad \text{for} \quad y > l, \end{aligned}\right\} \tag{23}$$

for the symmetrical disturbance, and

$$\left.\begin{aligned} \pi_- &= K_1[\exp(\tilde{\alpha}\xi_- y) + \exp(-\tilde{\alpha}\xi_- y)] \quad \text{for} \quad 0 < y < l, \\ \pi_+ &= K_1[\exp(\tilde{\alpha}\xi_- l) + \exp(-\tilde{\alpha}\xi_- l)]\exp[-\tilde{\alpha}\xi_+(y-l)] \quad \text{for} \quad y > l, \end{aligned}\right\} \tag{24}$$

for the antisymmetrical disturbance. The subscripts '+' and '−' are used to indicate quantities outside and inside the upper vortex sheet, respectively.

Let the displacement of the vortex sheet be

$$\eta = B \exp[i(\alpha x + \beta z - \alpha c t)], \tag{25}$$

where B is a constant. The rate of change of the displacement of the vortex sheet equals the vertical velocity disturbance at the sheet; therefore, from the equation of motion (4) it follows that

$$-\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial y} = \frac{D^2\eta}{Dt^2}, \tag{26}$$

where
$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x},$$

and p' is the pressure disturbance. With the aid of (26) and the pressure matching condition, the eigenvalue equations are given by

$$\bar{\rho}_+\xi_- \coth(\tilde{\alpha}\xi_- l)(U_+-c)^2 = -\bar{\rho}_-(U_--c)^2\xi_+, \tag{27}$$

for the symmetrical disturbance and

$$\bar{\rho}_+\xi_- \tanh(\tilde{\alpha}\xi_- l)(U_+-c)^2 = -\bar{\rho}_-(U_--c)^2\xi_+, \tag{28}$$

for the anti-symmetrical disturbance. For $\Theta = 0°$, these equations reduce to those for a two-dimensional disturbance obtained by Pai and Miles.

2.3. *Subsonic and supersonic disturbances*

A disturbance is considered subsonic, or supersonic, according to whether its wave speed relative to the velocity of the flow in the direction of wave propagation is less than, or greater than, the local sonic velocity. Therefore, from (20), ξ is real for subsonic disturbances, imaginary for supersonic ones and complex for amplified or damped ones.

Neutral subsonic disturbances relative to both the jet and the free stream cannot exist, because in this case the right-hand side of (27) or (28) is always negative, while the left-hand side of the equation is always positive.

When c is a complex number, i.e. the disturbance is amplified or damped, both (27) and (28) are complex equations. Each of these equations can be separated into two equations; one consists of the real part and the other the imaginary part of the original complex equation. It can be shown that changing the sign of c_i, when $|U_\pm - c| < a_\pm$, will not alter the separated equations (see Appendix). Thus c has the form

$$c = E \pm iF, \tag{29}$$

where E and F are functions of $\tilde{\alpha}l$, $\bar{\rho}_\pm$ and $\xi_\pm$. Since there are always positive c_i possible, the flow is unstable under subsonic disturbances.

It is interesting to note that neutral supersonic disturbances relative to the jet can exist, because ξ_- is purely imaginary for these disturbances, and (27) and (28) can be written as follows:

$$\bar{\rho}_+ |\xi_-| \cot(\tilde{\alpha}l|\xi_-|)(U_+ - c)^2 = \bar{\rho}_-(U_- - c)^2 \xi_+, \tag{30}$$

$$\bar{\rho}_+ |\xi_-| \tan(\tilde{\alpha}l|\xi_-|)(U_- - c)^2 = -\bar{\rho}_-(U_- - c)^2 \xi_+. \tag{31}$$

Cot$(\tilde{\alpha}l|\xi_-|)$ and $\tan(\tilde{\alpha}l|\xi_-|)$ are not single-valued. Therefore, the eigenvalue $\tilde{\alpha}$ may be any value, and (30) and (31) give a continuous spectrum for $\tilde{\alpha}$.

In the absence of neutral disturbances, the flow will be unstable when it is subjected to disturbances which are supersonic relative to the jet, but subsonic relative to the free stream (see Appendix).

3. Axially symmetric jet

3.1. *Disturbance equations*

Consider the inviscid flow of an axially symmetric jet or a cylindrical vortex sheet of a compressible fluid. Again, the jet flow is assumed to be parallel, so that the main flow has a uniform velocity U in the flow direction, say z, only. Suppose the flow is subjected to a small disturbance, and the velocity components in cylindrical polar co-ordinates are given by

$$u_z = U + u'_z, \quad u_r = u'_r, \quad u_\phi = u'_\phi, \tag{32}$$

where primes are used to denote disturbance quantities. Thus the linearized disturbance equations take the following form:

continuity $$\frac{\partial \rho'}{\partial t} + U\frac{\partial \rho'}{\partial z} = -\bar{\rho}\left[\frac{\partial u'_z}{\partial z} + \frac{1}{r}\frac{\partial(ru'_r)}{\partial r} + \frac{1}{r}\frac{\partial u'_\phi}{\partial \phi}\right], \tag{33}$$

momentum
$$\bar{\rho}\left(\frac{\partial u_z'}{\partial t}+U\frac{\partial u_z'}{\partial z}\right)=-\frac{\partial p'}{\partial z}, \tag{34}$$

$$\bar{\rho}\left(\frac{\partial u_r'}{\partial t}+U\frac{\partial u_r'}{\partial z}\right)=-\frac{\partial p'}{\partial z}, \tag{35}$$

$$\bar{\rho}\left(\frac{\partial u_\phi'}{\partial t}+U\frac{\partial u_\phi'}{\partial z}\right)=-\frac{1}{r}\frac{\partial p'}{\partial \phi}, \tag{36}$$

energy
$$\bar{\rho}c_v\left(\frac{\partial T'}{\partial t}+U\frac{\partial T'}{\partial t}\right)=-p\left(\frac{\partial u_r'}{\partial r}+\frac{u_r'}{r}+\frac{1}{r}\frac{\partial u_\phi'}{\partial \phi}+\frac{\partial u_z'}{\partial z}\right), \tag{37}$$

state
$$p'/\bar{p}=\rho'/\bar{\rho}+T'/\bar{T}. \tag{38}$$

Assume that the disturbance has sinusoidal dependence on both αz and $n\phi$, where α is the axial wave-number, and n is a positive integer. The disturbance is assumed to vary exponentially in time t. Thus, disturbance quantities may be written as follows:

$$\left.\begin{aligned} u_r', u_\phi', u_z' &= \{f(r), g(r), h(r)\}\exp\left[in\phi+i\alpha(z-ct)\right], \\ p', T', \rho' &= \{\pi(r), \theta(r), m(r)\}\exp\left[in\phi+i\alpha(z-ct)\right]. \end{aligned}\right\} \tag{39}$$

By introducing this type of disturbance, the disturbance equations become:†

continuity
$$i\alpha(U-c)\,m=-\bar{\rho}[i(\alpha h+ng/r)+f'+f/r], \tag{40}$$

momentum
$$\bar{\rho}(U-c)\,h=-\pi, \tag{41}$$

$$i\alpha\bar{\rho}(U-c)f=-\pi', \tag{42}$$

$$\alpha\bar{\rho}(U-c)\,g=-n\pi/r, \tag{43}$$

energy
$$i\alpha(U-c)\,\theta=-\bar{T}(\gamma-1)\,[i(\alpha h+ng/r)+f'+f/r], \tag{44}$$

state
$$\pi/\bar{p}=m/\bar{\rho}+\theta/\bar{T}. \tag{45}$$

Elimination of m, g, h and θ in (40), (41) and (43), (44), (45) in a manner similar to that used for the two-dimensional case gives

$$i\left[\frac{\alpha^2+n^2/r^2}{\alpha(U-c)\bar{\rho}}-\frac{\alpha}{\gamma\bar{p}}(U-c)\right]\pi=f'+\frac{f}{r}. \tag{46}$$

From (42) and (45), the following expression for the pressure disturbance is obtained:

$$\pi''+\frac{1}{r}\pi'+\left\{\alpha^2\left[\left(\frac{U-c}{a}\right)^2-1\right]-\frac{n^2}{r^2}\right\}\pi=0, \tag{47}$$

where a is the local sonic velocity. For $a\to\infty$, this equation reduces to that for incompressible flow obtained by Batchelor & Gill (1962).

The asymptotic behaviour of π is

$$\pi\sim\exp\left\{-\alpha r\sqrt{\left[1-\left(\frac{U(\infty)-c}{a}\right)^2\right]}\right\}\quad\text{for}\quad r\to\infty. \tag{48}$$

Again, it is assumed that the flow is isoenergetic, so that equation (47) holds for both inside and outside of the jet.

† For the remainder of this section we use 'prime' to denote differentiation with respect to r, unless otherwise specified.

3.2. *Stability considerations*

For neutral subsonic disturbances relative to both the jet and the free stream, c is real, and

$$\left(\frac{U_\pm - c}{a_\pm}\right)^2 < 1,$$

where '$-$' and '$+$' denote quantities inside and outside the vortex sheet, respectively. Therefore, equation (47) becomes a modified Bessel's equation, and the solution for the pressure disturbance is given by

$$\pi = C_1 I_n(\alpha\xi r) + C_2 K_n(\alpha\xi r), \tag{49}$$

where

$$\xi = \sqrt{\left[1 - \left(\frac{U-c}{a}\right)^2\right]},$$

C_1 and C_2 are constants, and I_n and K_n are modified Bessel's functions of the first and the second kinds, respectively. Because I_n is unbounded at infinity, and K_n at $r = 0$, it follows that

$$\left.\begin{aligned} \pi_- &= C_1 I_n(\alpha\xi_- r) \quad \text{for} \quad r < r_0, \\ \pi_+ &= C_2 K_n(\alpha\xi_+ r) \quad \text{for} \quad r > r_0, \end{aligned}\right\} \tag{50}$$

where r_0 is the radius of the undisturbed vortex sheet.

Let the displacement of the vortex sheet be

$$\eta = B \exp\left[in\phi + i\alpha(z - ct)\right], \tag{51}$$

where B is a constant. The rate of change of the displacement of the vortex sheet equals the radial velocity disturbance at the sheet; therefore, from (35) we have

$$-\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial r} = \frac{D^2\eta}{Dt^2}, \tag{52}$$

where

$$\frac{D}{Dt^2} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial z},$$

and p' is the pressure disturbance. Assume that mean pressure in the jet and the free stream are equal. From equation (52), with the help of (50), (51) and the matching condition that the pressure disturbances on both sides of the vortex sheet are equal, the following eigenvalue equation is obtained:

$$\left(\frac{U_- - c}{U_+ - c}\right)^2 = \frac{\bar{\rho}_+}{\bar{\rho}_-}\frac{K_n(\alpha\xi_+ r_0)\, I_n'(\alpha\xi_- r_0)}{K_n'(\alpha\xi_+ r_0)\, I_n(\alpha\xi_- r_0)}. \tag{53}$$

For real c, only real and positive I_n, I_n' and K_n are obtained, while K_n' is real and negative; therefore, (53) can be written as

$$\left(\frac{U_- - c}{U_+ - c}\right)^2 = -L_n(\alpha r_0, \xi_\pm, \bar{\rho}_\pm), \tag{54}$$

where $L_n(\alpha r_0, \xi_\pm, \bar{\rho}_\pm)$ is a real positive number. Obviously, both sides of (54) are inconsistent; therefore, this kind of disturbance does not exist in the inviscid flow of a cylindrical vortex sheet.

3.2.1. *Neutral subsonic and supersonic disturbances.* When disturbances are supersonic relative to the jet, but subsonic relative to the free stream, (47) becomes a Bessel equation inside the jet. Since Bessel functions of the second kind are unbounded at $r = 0$, the disturbance solutions are given by

$$\left.\begin{aligned} \pi_- &= C_1 J_n(\alpha\zeta_- r) \quad \text{for} \quad r < r_0, \\ \pi_+ &= C_2 K_n(\alpha\xi_+ r) \quad \text{for} \quad r > r_0, \end{aligned}\right\} \tag{55}$$

where
$$\zeta_- = \sqrt{\left[\left(\frac{U_- - c}{a}\right)^2 - 1\right]}.$$

The eigenvalue equation is then obtained by simply replacing I_n and I'_n in (53) by J_n and J'_n respectively. The equation is

$$\left(\frac{U_- - c}{U_+ - c}\right)^2 = \frac{\bar{\rho}_+}{\bar{\rho}_-} \frac{K_n(\alpha\xi_+ r_0)\, J'_n(\alpha\zeta_- r_0)}{K'_n(\alpha\xi_+ r_0)\, J_n(\alpha\zeta_- r_0)}. \tag{56}$$

K_n/K'_n is always negative for a real argument. Because of the oscillatory behaviour of J_n, J'_n/J_n can be negative, and equation (56) gives a continuous spectrum for α.

3.2.2. *Amplified or damped disturbances.* In this case, c, and hence the arguments of Bessel and modified Bessel functions, are complex. Solutions of the type given by equation (50) are chosen. The reason for choosing K_n for $r > r_0$ is that some values of Y_n and J_n are unbounded when their arguments are in the first or the fourth complex quadrant. Thus, solutions for π are

$$\left.\begin{aligned} \pi_- &= C_1 I_n(\alpha\xi_- r) \quad \text{for} \quad r < r_0, \\ \pi_+ &= C_2 K_n(\alpha\xi_+ r) \quad \text{for} \quad r > r_0, \end{aligned}\right\} \tag{57}$$

where C_1 and C_2 are complex constants.

The eigenvalue equation is the same as (53), but this becomes complex.

4. Numerical results and conclusions

The foregoing analysis can also be applied to the stability of the inviscid wakes, since no restriction has been made on the main flow velocities.

For the two-dimensional jet, numerical calculations have been performed for the case of a jet issuing from a nozzle into a stationary surrounding, $U_+ = 0$, for both symmetrical and antisymmetrical disturbances. Results are plotted in figures 1–6. From figures 4 and 6, as Mach number increases, the amplification factor of the disturbances c_i decreases except for very small wave-number. Thus, the increasing of the Mach number of the jet tends to stabilize the flow. There is no upper limit in Mach number, as was found for a plane vortex sheet, above which the flow is completely stable (Pai 1954; Miles 1958).

The angle of wave propagation also influences the stability of the flow. Figures 1–3 and 5 show that when the angle between the direction of the wave propagation and that of the jet flow increases, c_i increases except for very small α. Thus, the increasing of the wave propagation angle tends to destabilize the flow.

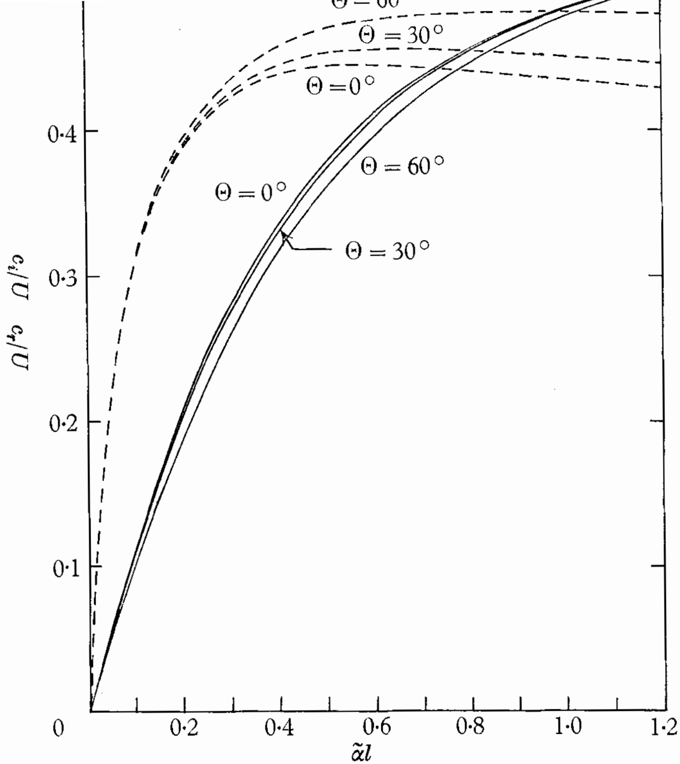

FIGURE 1. Wave speed c_r (solid lines) and amplification factor c_i (broken lines) for a symmetrical disturbance to a two-dimensional jet, $n = 1$.

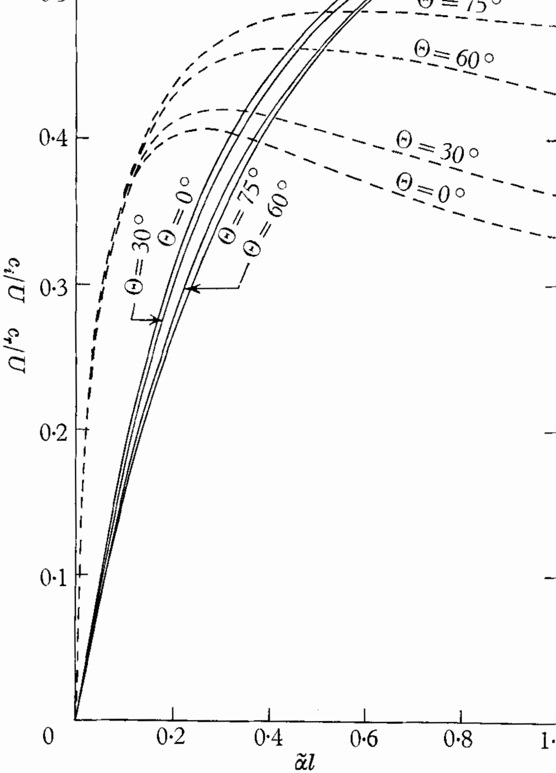

FIGURE 2. Wave speed c_r (solid lines) and ampli cation factor c_i (broken lines) for a symmetric disturbance to a two-dimensional jet, $M = 2$.

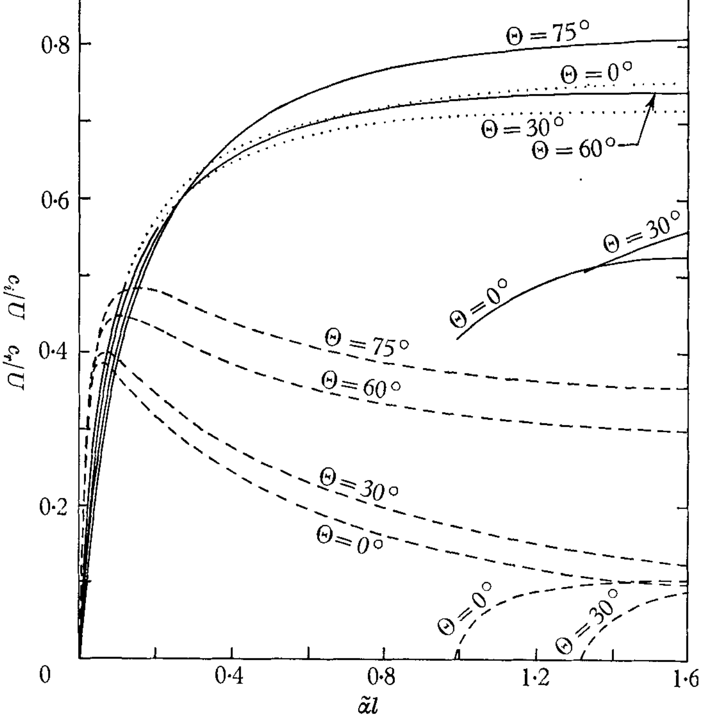

FIGURE 3. Wave speed c_r (solid or dotted lines) and amplification factor c_i (broken lines) for a symmetrical disturbance to a two-dimensional jet, $M = 5$.

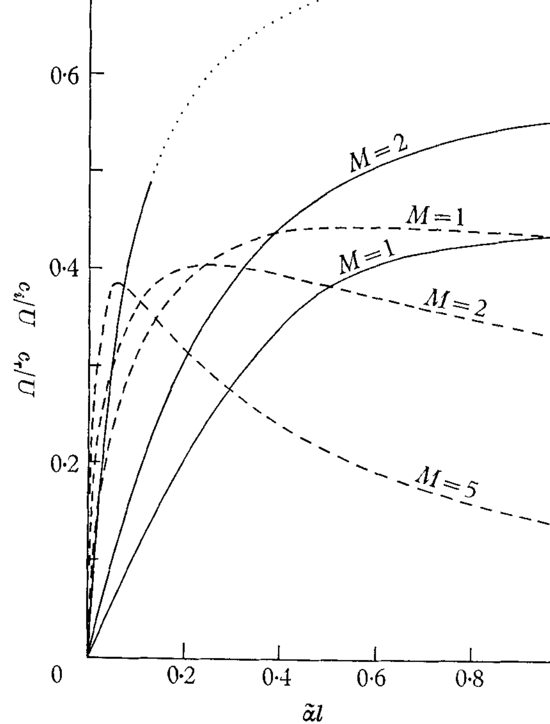

FIGURE 4. Wave speed c_r (solid or dotted lines) a amplification factor c_i (broken lines) for a sy metrical disturbance to a two-dimensional jet, Θ =

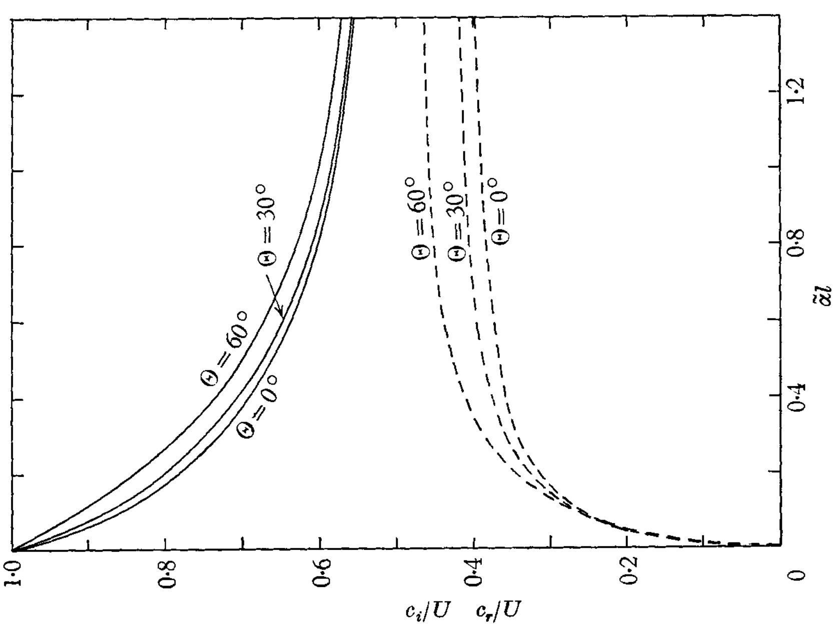

FIGURE 5. Wave speed c_r (solid lines) and amplification factor c_i (broken lines) for an antisymmetrical disturbance to a two-dimensional jet, $M = 1$.

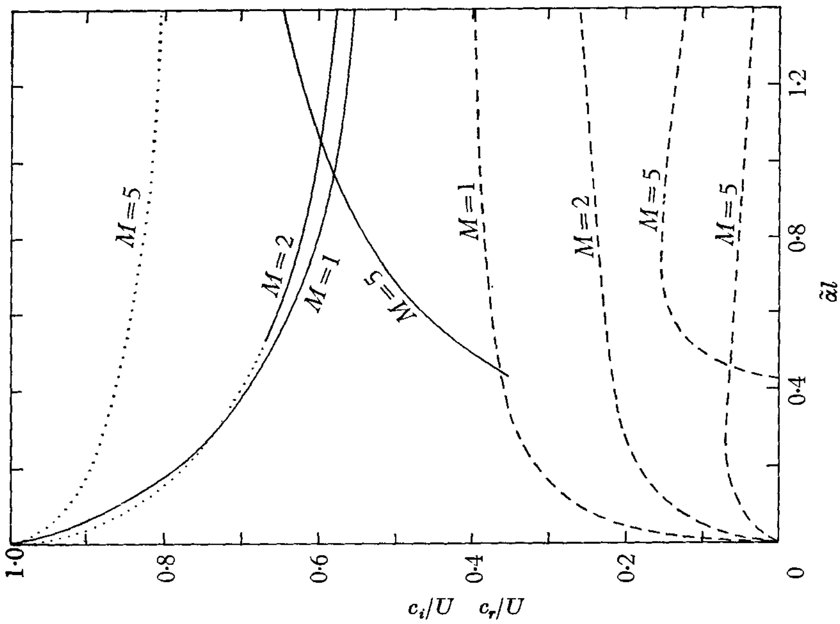

FIGURE 6. Wave speed c_r (solid or dotted lines) and amplification factor c_i (broken lines) for an antisymmetrical disturbance to a two-dimensional jet, $\Theta = 0°$.

It has been made clear by this calculation that supersonic disturbances inside the jet may cause instability. When $\Theta = 0°$, no eigenvalue has been found for the case with supersonic disturbances inside the jet for Mach number M unity (figure 1). As the Mach number is increased to 2, eigenvalues corresponding to both supersonic and subsonic disturbances inside the jet are obtained. At even higher Mach number, all the eigenvalues correspond to supersonic disturbances relative to the jet.

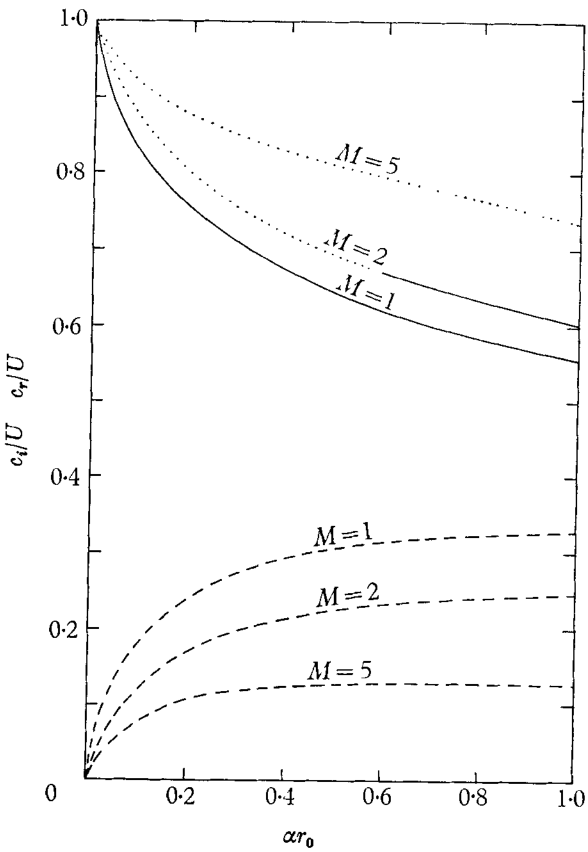

FIGURE 7. Wave speed c_r (solid or dotted lines) and amplification factor c_i for an axially symmetrical disturbance, $n = 0$, to an axially symmetric jet.

Physically, when the disturbance is supersonic inside the jet, there can be acoustic energy transfer from the vortex sheet into the jet. However, if the disturbance is subsonic outside the jet, the energy wave inside the jet is reflected at the vortex sheet. When the disturbance is also supersonic outside the jet, acoustic energy radiation between the jet and the surrounding fluid occurs. If the radiation is small enough, the disturbance may still be destabilizing. Should this be the case, the dotted curves in figures 3, 4, 6 and 7 give the eigenvalues for supersonic disturbances outside the jet which may also cause instability, but the jet is more stable to these than to the subsonic disturbances.

As far as supersonic disturbances are concerned, either inside or outside the jet, there will be more than one mode of instability, as it is clear from figures 3 and 6; however, it is expected that the mode corresponding to the smaller wave-number is more unstable.

For the axially symmetric jet, since no prediction can be made analytically, numerical calculations for the case of the jet issuing from the nozzle have been carried out. The cases $n = 0$ and $n = 1$ have been studied. The eigenvalue curves

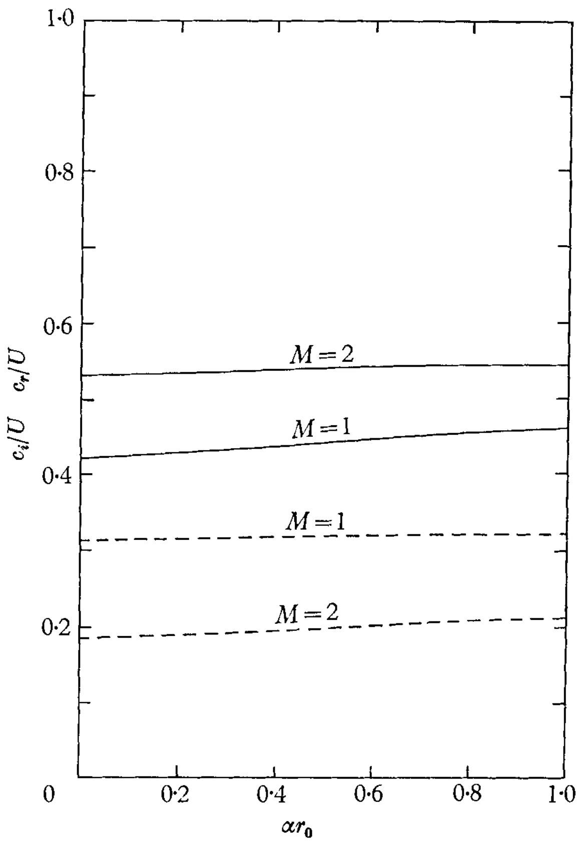

FIGURE 8. Wave speed c_r (solid lines) and amplification factor c_i (broken lines) for an azimuthally periodic disturbance, $n = 1$, to an axially symmetric jet.

are similar to those for incompressible jets obtained by Batchelor & Gill (1962). It also shows that the flow tends to be less unstable when the Mach number is increased. Although calculations have not been made for the case $n > 1$, it can be seen from figures (7) and (8), as compared with the results for incompressible flow, that azimuthally periodic disturbances will be more unstable than the rotationally symmetric ones only at small axial wave-number.

Appendix

Amplified or damped disturbances for a two-dimensional jet

Consider subsonic disturbances relative to both the two-dimensional jet and the free stream. In this case, c is a complex number. Let qU be the mean velocity of the free stream, where q may be greater or less than unity. Equation (27) can be written in dimensionless form

$$\left(\frac{1-c^*}{c^*}\right)^2 = -\sqrt{\left(\frac{1-\tilde{M}^2(1-c^*)^2}{1-\tilde{M}^2(q-c^*)^2/\bar{T}^*}\right)}\frac{\coth(\tilde{\alpha}^*\sqrt{\{1-\tilde{M}^2(1-c^*)^2\}})}{\bar{T}^*}, \quad \text{(A 1)}$$

where
$$c^* = c/U_- = c_r^* + ic_i^*,$$
$$\tilde{\alpha}^* = \tilde{\alpha}l,$$
$$\tilde{M} = M_0\cos\Theta,$$
and
$$\bar{T}^* = \bar{T}_+/\bar{T}_- = 1+\tfrac{1}{2}(\gamma-1)M_0^2(1-q^2)$$

for isoenergetic flow. M_0 is the Mach number of the jet. The rearranging of (A 1) yields

$$\bar{T}^*(1-c^*)^2\sqrt{\left[1-\frac{\tilde{M}^2(q-c^*)^2}{\bar{T}^*}\right]} = (q-c^*)^2\sqrt{[1-\tilde{M}^2(1-c^*)^2]}$$
$$\times\coth\{\tilde{\alpha}^*\sqrt{[1-\tilde{M}^2(1-c^*)^2]}\}. \quad \text{(A 2)}$$

The real part of the above equation is

$$-\bar{T}^*\{[(1-c_r^*)^2-c_i^{*2}]\mathscr{R}+2(1-c_r^*)c_i^*\mathscr{S}\}$$
$$= [(q-c_r^*)^2-c_i^*](\mathscr{H}\mathscr{M}-\mathscr{K}\mathscr{N})-2c_i^*(q-c_r^*)(\mathscr{K}\mathscr{M}+\mathscr{H}\mathscr{N}), \quad \text{(A 3)}$$

and the imaginary part is

$$-\bar{T}^*\{[(1-c_r^*)^2-c_i^{*2}]\mathscr{S}-2(1-c_r^*)c_i^*\mathscr{R}\}$$
$$= 2c_i^*(q-c_r^*)(\mathscr{H}\mathscr{M}-\mathscr{K}\mathscr{N})+[(q-c_r^*)^2-c_i^{*2}](\mathscr{K}\mathscr{M}+\mathscr{H}\mathscr{N}). \quad \text{(A 4)}$$

where
$$\mathscr{R} = \frac{1}{\sqrt{2}}\sqrt{\left(1-\frac{\tilde{M}^2}{\bar{T}^*}[(q-c_r^*)^2-c_i^{*2}]\right.}$$
$$\left.+\sqrt{\left(\left\{1-\frac{\tilde{M}^2}{\bar{T}^*}[(q-c_r^*)^2-c_i^{*2}]\right\}^2+4\frac{\tilde{M}^4}{\bar{T}^{*2}}(q-c_r^*)^2c_i^{*2}\right)}\right),$$

$$\mathscr{S} = \frac{1}{\mathscr{R}}\frac{\tilde{M}^2}{\bar{T}^*}(q-c_r^*)c_i^*,$$

$$\mathscr{H} = \frac{1}{\sqrt{2}}\sqrt{(1-\tilde{M}^2[(1-c_r^*)^2-c_i^{*2}]}$$
$$+\sqrt{(\{1-\tilde{M}^2[(1-c_r^*)^2-c_i^{*2}]\}^2+4c_i^{*2}(1-c_r^*)^2\tilde{M}^4))},$$

$$\mathscr{K} = \frac{c_i^*(1-c_r^*)\tilde{M}^2}{\mathscr{H}},$$

$$\mathscr{M} = \frac{\sinh 2\tilde{\alpha}^*\mathscr{H}}{\cosh 2\tilde{\alpha}^*\mathscr{H}-\cos 2\tilde{\alpha}^*\mathscr{K}},$$

$$\mathscr{N} = \frac{-\sin 2\tilde{\alpha}^*\mathscr{K}}{\cosh 2\tilde{\alpha}^*\mathscr{H}-\cos 2\tilde{\alpha}^*\mathscr{K}}.$$

Note that $\mathscr{R}$, $\mathscr{H}$ and $\mathscr{M}$ are even in c_i and $\mathscr{S}$, $\mathscr{K}$ and $\mathscr{N}$ are odd in c_i. Thus, changing the sign of c_i will not alter equations (A 3) and (A 4), and c takes the form

$$c = E \pm iF, \tag{29}$$

where E and F are functions of $\tilde{\alpha}l$, $\bar{\rho}_{\pm}$ and $\xi_{\pm}$.

For disturbances which are supersonic relative to the free stream, (A 3) and (A 4) can still be used. These equations also remain unchanged by changing the sign of c_i.

If $\mathscr{M}$ and $\mathscr{N}$ in (A 3) and (A 4) are replaced by

$$\mathscr{M}/(\mathscr{M}^2+\mathscr{N}^2) \quad \text{and} \quad -\mathscr{N}/(\mathscr{M}^2+\mathscr{N}^2),$$

respectively, it is immediately proved that the above result is also true for antisymmetrical disturbances.

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